EXTREMAL FUNCTIONS OF BOUNDARY SCHWARZ LEMMA

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ABSTRACT. In this paper, we present an alternative and elementary proof of a sharp version of the classical boundary Schwarz lemma by Frolova et al. with initial proof via analytic semigroup approach and Julia-Carathéodory theorem for univalent holomorphic self-mappings of the open unit disk $\mathbb{D} \subset \mathbb{C}$. Our approach has its extra advantage to get the extremal functions of the inequality in the boundary Schwarz lemma.

1. Introduction

The Schwarz lemma as one of the most influential results in complex analysis puts a great push to the development of several research fields, such as geometric function theory, hyperbolic geometry, complex dynamical systems, composition operators theory, and theory of quasiconformal mappings. We refer to [1, 4] for a more complete insight on the Schwarz lemma.

The classical Schwarz lemma as well as the Schwarz-Pick lemma concerns with holomorphic self-mappings of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, which provides the invariance of the hyperbolic disks around the interior fixed point under self-mappings of $\mathbb{D}$. A verity of its boundary versions are in the spirit of Julia [7], Carathéodory [3], and Wolff [14, 15] involving the boundary involving the boundary fixed points.

Recall that a boundary point $\xi \in \partial\mathbb{D}$ is called a fixed point of $f \in \mathsf{H}(\mathbb{D}, \mathbb{D})$ if

$$f(\xi) := \lim_{r \to 1^-} f(r\xi) = \xi.$$

Here $\mathsf{H}(\mathbb{D}, \mathbb{D})$ denotes the class of holomorphic self-mappings of the open unit disk $\mathbb{D}$. It is well known that, for any $f \in H(\mathbb{D}, \mathbb{D})$, its radial limit is the same as its angular limit and both exist for almost all $\xi \in \partial\mathbb{D}$; moreover, the exceptional set in $\partial\mathbb{D}$ is of capacity zero.

The classification of the boundary fixed points of $f \in \mathsf{H}(\mathbb{D}, \mathbb{D})$ can be performed via the value of the *angular derivative*

$$f'(\xi) := \angle \lim_{z \to \xi} \frac{f(z) - \xi}{z - \xi},$$

which belongs to $(0, \infty]$ due to the celebrated Julia-Carathéodory theorem; see [3, 1]. This theorem also asserts that the finite angular derivative at the boundary fixed point ξ exists if and only if the holomorphic function f' admits the finite angular limit $\angle \lim_{z \to \xi} f'(z)$. For a boundary fixed point ξ of f, if

$$f'(\xi) \in (0, \infty),$$

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then ξ is called a regular boundary fixed point. The regular points can be *attractive* if $f'(\xi) \in (0,1)$, *neutral* if $f'(\xi) = 1$, or *repulsive* if $f'(\xi) \in (1,\infty)$.

The Julia-Carathéodory theorem [3, 1] and the Wolff lemma [15] imply that there exists a unique regular boundary fixed point ξ such that

$$f'(\xi) \in (0,1]$$

if $f \in \mathsf{H}(\mathbb{D},\mathbb{D})$ with no interior fixed point; otherwise the assumption that the mapping $f \in \mathsf{H}(\mathbb{D},\mathbb{D})$ with an interior fixed point forces $f'(\xi) > 1$ for any boundary fixed point $\xi \in \partial\mathbb{D}$. Moreover, Unkelbach [13] and Herzig [6] proved that if $f \in \mathsf{H}(\mathbb{D},\mathbb{D})$ has a regular boundary fixed point at point 1, and $f(0) = 0$, then

$$f'(1) \geq \frac{2}{1+|f'(0)|}. \tag{1.1}$$

Moreover, equality in (1.1) holds if and only if f is of the form

$$f(z) = -z\frac{a-z}{1-az}, \qquad \forall\, z \in \mathbb{D},$$

for some constant $a \in (-1,0]$.

This result is improved sixty years later by Osserman [9] by removing the assumption of the existence of interior fixed points.

Theorem 1.1. (Osserman) *Let $f \in H(\mathbb{D},\mathbb{D})$ with $\xi = 1$ as its regular boundary fixed point. Then*

$$f'(1) \geq \frac{2\big(1-|f(0)|\big)^2}{1-|f(0)|^2+|f'(0)|}. \tag{1.2}$$

This inequality is strengthened very recently by Frolova et al. in [5] as follows.

Theorem 1.2. (Frolova et al.) *Let $f \in H(\mathbb{D},\mathbb{D})$ with $\xi = 1$ as its regular boundary fixed point. Then*

$$f'(1) \geq \frac{2}{\mathrm{Re}\left(\dfrac{1-f(0)^2+f'(0)}{(1-f(0))^2}\right)}. \tag{1.3}$$

The initial proof given in [5] is based on analytic semigroup approach as well as the Julia-Carathéodory theorem for *univalent* holomorphic self-mappings of $\mathbb{D}$, which is proved via the method of extremal length [2].

The purpose of this article is to study the extremal functions of inequality (1.3), in addition to present an alternative and elementary proof of (1.3). Our main result is as follows.

Theorem 1.3. *Let $f \in H(\mathbb{D},\mathbb{D})$ with $\xi = 1$ as its regular boundary fixed point. Then equality holds in inequality* (1.3) *if and only if f is of the form*

$$f(z) = \frac{f(0) - z\dfrac{a-z}{1-az}\dfrac{1-f(0)}{1-\overline{f(0)}}}{1 - z\dfrac{a-z}{1-az}\dfrac{1-f(0)}{1-\overline{f(0)}}\overline{f(0)}}, \qquad \forall\, z \in \mathbb{D}, \tag{1.4}$$

for some constant $a \in [-1,1)$.

As a direct consequence of Theorems 1.2 and 1.3, we obtain a strong version of Osserman's inequality.

Corollary 1.4. *Let $f \in H(\mathbb{D},\mathbb{D})$ with $\xi = 1$ as its regular boundary fixed point. Then*

$$f'(1) \geq \frac{2\left|1-f(0)\right|^2}{1-|f(0)|^2+|f'(0)|}. \tag{1.5}$$

Moreover, equality holds in this inequality if and only if f is of the form

$$f(z) = \frac{f(0) - z\dfrac{a-z}{1-az}\dfrac{1-f(0)}{1-\overline{f(0)}}}{1 - z\dfrac{a-z}{1-az}\dfrac{1-f(0)}{1-\overline{f(0)}}\overline{f(0)}}, \qquad \forall\, z \in \mathbb{D}, \tag{1.6}$$

for some constant $a \in [-1, 0]$.

Remark 1.5. From Corollary 1.4, one easily deduce that equality in (1.1) hold if and only if f is of the form

$$f(z) = \frac{f(0) - z\dfrac{a-z}{1-az}}{1 - z\dfrac{a-z}{1-az}f(0)}, \qquad \forall\, z \in \mathbb{D},$$

for some constant $a \in [-1, 0]$ with $f(0) \in [0, 1)$.

As an application, Corollary 1.4 immediately results in a quantitative strengthening of a classical theorem of Löwner (i.e the second assertion in the following Corollary, see [8]) as follows.

Corollary 1.6. (Löwner) *Let $f \in H(\mathbb{D},\mathbb{D})$ and $f(0) = 0$. Assume that f extends continuously to an arc $C \in \partial\mathbb{D}$ of length s and maps it onto an arc $f(C) \in \partial\mathbb{D}$ of length σ. Then*

$$\sigma \geq \frac{2}{1+|f'(0)|}s.$$

In particular, we have

$$\sigma \geq s \tag{1.7}$$

with equality if and only if either $\sigma = s = 0$ or f is just a rotation.

Remark 1.7. The length σ of $f(C)$ is to be taken with multiplicity, if $f(C)$ is a multiple covering of the image.

For generalizations of Theorems 1.2 and 1.3 as well as Corollary 1.4 to the setting of quaternions for slice regular self-mappings of the open unit ball $\mathbb{B} \in \mathbb{H}$, see [10].

2. Proof of main results

In this section, we shall give the proofs of the main results. Before presenting the details, we first recall the concrete contents of the classical Julia lemma and Julia-Carathéodory theorem; see e.g. [1, 11], [12, p. 48 and p. 51].

Lemma 2.1. (Julia) *Let $f \in H(\mathbb{D},\mathbb{D})$ and let $\xi \in \partial\mathbb{D}$. Suppose that there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{D}$ converging to ξ as n tends to ∞, such that the limits*

$$\alpha := \lim_{n\to\infty} \frac{1-|f(z_n)|}{1-|z_n|}$$

and

$$\eta := \lim_{n\to\infty} f(z_n)$$

exist (*finitely*). *Then* $\alpha > 0$ *and the inequality*

$$\frac{\big|f(z)-\eta\big|^2}{1-|f(z)|^2} \le \alpha\, \frac{|z-\xi|^2}{1-|z|^2} \tag{2.1}$$

holds throughout the open unit disk $\mathbb{D}$ *and is strict except for Möbius transformations of* $\mathbb{D}$.

Theorem 2.2. (Julia-Carathéodory) *Let* $f \in \mathsf{H}(\mathbb{D},\mathbb{D})$ *and let* $\xi \in \partial\mathbb{D}$. *Then the following conditions are equivalent:*

(i) *The lower limit*

$$\alpha := \liminf_{z\to\xi} \frac{1-|f(z)|}{1-|z|} \tag{2.2}$$

is finite, where the limit is taken as z *approaches* ξ *unrestrictedly in* $\mathbb{D}$;

(ii) f *has a non-tangential limit, say* $f(\xi)$, *at the point* ξ, *and the difference quotient*

$$\frac{f(z)-f(\xi)}{z-\xi}$$

has a non-tangential limit, say $f'(\xi)$, *at the point* ξ;

(iii) *The derivative* f' *has a non-tangential limit, say* $f'(\xi)$, *at the point* ξ.

Moreover, under the above conditions we have

(a) $\alpha > 0$ *in* (i);

(b) *the derivatives* $f'(\xi)$ *in* (ii) *and* (iii) *are the same;*

(c) $f'(\xi) = \alpha\overline{\xi}f(\xi)$;

(d) *the quotient* $\dfrac{1-|f(z)|}{1-|z|}$ *has the non-tangential limit* α *at the point* ξ.

Now we come to prove Theorems 1.2 and 1.3.

Proofs of Theorems 1.2 *and* 1.3. Let f be as described in Theorem 1.2. Set

$$g(z) := \frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}\,\frac{1-\overline{f(0)}}{1-f(0)},$$

which is in $\mathsf{H}(\mathbb{D},\mathbb{D})$ such that $\xi = 1$ is its regular boundary fixed point and $g(0) = 0$. Moreover, an easy calculation shows that

$$f'(1) = \frac{|1-f(0)|^2}{1-|f(0)|^2}\, g'(1), \tag{2.3}$$

and

$$g'(0) = \frac{f'(0)}{1-|f(0)|^2}\,\frac{1-\overline{f(0)}}{1-f(0)}, \tag{2.4}$$

which is no more than one in modulus. Applying the Julia-Carathéodory theorem and the Julia inequality (2.1) in the Julia lemma to the holomorphic function $h : \mathbb{D} \to \overline{\mathbb{D}}$ defined by

$$h(z) := \frac{g(z)}{z}, \qquad \forall\, z \in \mathbb{D},$$

we obtain

$$g'(1) = 1 + h'(1) \geq 1 + \frac{|1 - g'(0)|^2}{1 - |g'(0)|^2} = \frac{2\big(1 - \mathrm{Re} g'(0)\big)}{1 - |g'(0)|^2}. \tag{2.5}$$

In particular,

$$g'(1) \geq \frac{2}{1 + \mathrm{Re} g'(0)}. \tag{2.6}$$

Now inequality (1.3) follows by substituting equalities in (2.3) and (2.4) into (2.6).

If equality holds in inequality (1.3), then equalities also hold in the Julia inequality (2.1) at point $z = 0$ and inequality (2.6), it follows from the condition for equality in the Julia inequality and that for equality in inequality (2.6) that

$$g(z) = z \frac{z - a}{1 - \bar{a}z} \frac{1 - \bar{a}}{1 - a}, \tag{2.7}$$

for some constant $a \in \overline{\mathbb{D}}$, and $g'(0) \in (-1, 1]$, which is possible only if $a \in [-1, 1)$. Consequently, f must be of the form

$$f(z) = \frac{f(0) - z\dfrac{a - z}{1 - az}\dfrac{1 - f(0)}{1 - \overline{f(0)}}}{1 - z\dfrac{a - z}{1 - az}\dfrac{1 - f(0)}{1 - \overline{f(0)}}\overline{f(0)}}, \qquad \forall\, z \in \mathbb{D}, \tag{2.8}$$

where $a \in [-1, 1)$. Therefore, the equality in inequality (1.3) can hold only for holomorphic self-mappings of the form (2.8), and a direct calculation shows that it does indeed hold for all such holomorphic self-mappings. This completes the proof. $\square$

Proof of Corollary 1.4. Inequality (1.5) follows immediately from inequality (1.3), and equality in (1.5) holds if and only if

$$\frac{f'(0)}{\big(1 - f(0)\big)^2} \in [0, \infty),$$

which is equivalent to $g'(0) \in [0, 1]$, i.e. $a \in [-1, 0]$. Here the function g is the one in (2.7).

$\square$

Proof of Corollary 1.6. By the classical Schwarz reflection principle, f can extend to be holomorphic in the interior of C. Applying Corollary 1.4 to the holomorphic self-mapping of $\mathbb{D}$ defined by

$$g(z) = \frac{f(\xi z)}{f(\xi)}, \qquad \forall\, z \in \mathbb{D}$$

yields the inequality

$$|f'(\xi)| \geq \frac{2}{1 + |f'(0)|}$$

for any point ξ in the interior of C. Consequently, the desired result follows. $\square$

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